

A NEW LOOK AT ADAPTED COMPLEX STRUCTURES

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R. Sz. dedicates this paper to his son, Bálint

ABSTRACT. Given a closed real analytic Riemannian manifold, we construct and study a one parameter family of adapted complex structures on the manifold of its geodesics.

1. Introduction. Adapted complex structures, also called Grauert tubes, are Kähler structures on the tangent or cotangent bundle of a Riemannian manifold M . They first appeared in [GS] and [LSz1]. Let $T^r M \subset TM$ denote the set of tangent vectors of length $< r$. By the above papers and by [L, Sz1] one knows that for a closed M an adapted complex structure exists on $T^r M$ for some $r > 0$ if and only if the metric of M is analytic; moreover, when it exists, the adapted complex structure is unique. These results have lately been extended to Koszul manifolds, i.e., manifolds with a connection, in [B, Sz3].

The thesis of this note is that one can talk about adapted complex and related structures in generality much greater than that of Riemannian or Koszul manifolds, indeed whenever a Lie semigroup acts on a manifold. Of course, whether these more general structures exist, are unique, or have useful properties, depends on the action. Our focus here will still be on Riemannian manifolds. For them the new definition is not quite equivalent to the one in [GS, LSz1]. Rather, there is a family of equally natural Kähler structures on (subdomains of) TM , parametrized by $s \in \mathbb{C} \setminus \mathbb{R}$. The Kähler manifolds thus obtained constitute the fibers of a holomorphic fibration over $\mathbb{C} \setminus \mathbb{R}$, and the adapted complex structure of [GS, LSz1] corresponds to the fiber over $s = i$.—It is possible to extend the fibration to a fibration over \mathbb{C} ; however, the fibers over \mathbb{R} will be real polarized rather than Kähler. Thus one is led to the notion of adapted polarizations, of which an adapted complex structure is just an extreme example.

The idea that the adapted complex structures of [GS, LSz1] give rise to other Kähler structures as well is as old as the whole subject. More to the point, the papers [FMMN1–2] consider a one (real) parameter family of Kähler structures on the cotangent bundle of a compact Lie group, that degenerates to a real polarization;

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this family is then used to explain geometrically the so called Bargmann–Segal–Hall transformation of [H1-2]. The papers themselves make no explicit connection with adapted complex structures, but the family considered there is the restriction of our family of adapted polarizations to the positive imaginary axis. Rather recently [HK] pointed out that for a general closed real analytic Riemannian manifold the original adapted complex structure is the analytic continuation to i of a real family of real polarizations. Undoubtedly, the authors were aware that continuation to other values $s \in \mathbb{C} \setminus \mathbb{R}$ also yields Kähler structures.

The novelty of the present note is first that all those Kähler structures and real polarizations can be derived from one principle; second that these structures, taken together, constitute a fiber bundle. The results obtained will be used in the companion paper [LSz2] to study the uniqueness of geometric quantization.

2. Polarized manifolds. A polarization of a smooth manifold N is given by a smooth, involutive, complex subbundle $P \subset \mathbb{C} \otimes TN$, of rank $m = (1/2) \dim N$. Involutivity means that the bracket of sections of P is again a section of P . This definition is more general than the one, say, in [W], but in our context this is the natural one. Sometimes even more general structures have to be considered, where the rank condition is omitted; these are the involutive manifolds.

A polarization is real if $\overline{P} = P$; it is equivalent to the datum of an m dimensional foliation of N . A polarization is complex if P and \overline{P} are transverse; this one is equivalent to a complex structure on N . In the former case P consists of tangents to the leaves, in the latter P is the bundle of $(1, 0)$ vectors. A smooth map f of polarized manifolds $(M, Q) \rightarrow (N, P)$ is called polarized if $f_*Q \subset P$.

Consider now a smooth manifold N on which a Lie semigroup G acts smoothly on the right. Fix a polarization of G .

Definition 1. A polarization of N is called adapted (to the polarization of G) if for every $x \in N$ the map $G \ni g \mapsto xg \in N$ is polarized.

All of the above makes sense and works the same when N is a manifold with boundary, except that the connection between real polarizations and foliations becomes tenuous (for there is more than one way to define foliations on such N).

3. The affine semigroup. This is the Lie semigroup G of affine transformations $\mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto gt = a + bt$. Here $a = a(g), b = b(g)$ serve as global coordinates on G , and identify it with \mathbb{R}^2 . Denote by L_g, R_g left and right translations of G . By a left invariant polarization Q of G we mean one for which $L_g: G \rightarrow G$ is polarized for every $g \in G$. In the identification $G \cong \mathbb{R}^2$ invariance means that the fibers Q_g , $g \in G$, are Euclidean translates of each other. Hence associating with a polarization Q of G the complex line $Q_{\text{id}} \subset \mathbb{C} \otimes T_{\text{id}}G$ yields a bijection between the set of left invariant polarizations and $\mathbb{P}(\mathbb{C} \otimes T_{\text{id}}G) \approx \mathbb{CP}_1$. All left invariant polarizations but one can be obtained by the following construction. The (left) action of G on \mathbb{R} extends to an affine action on \mathbb{C} . Fix $s \in \mathbb{C}$, let

$$f_s(g) = gs \quad \text{and} \quad Q(s) = (f_{s*})^{-1}T^{1,0}\mathbb{C}. \quad (1)$$

E.g., $Q(i)$ gives the usual complex structure on $G \cong \mathbb{R}^2$, while for s real $Q(s)$ is a real polarization whose leaves are straight lines of slope $-1/s$. Equivalently, with

$H_s = \{g \in G : gs = s\}$, $s \in \mathbb{R}$, the leaves of $Q(s)$ are left translates of H_s . We write $Q(\infty)$ for the exceptional, real polarization, whose leaves have slope 0. Since $f_{gs} = f_s \circ R_g$, (1) implies that $R_g : (G, Q(gs)) \rightarrow (G, Q(s))$ is a polarized map.

Define the character $\chi : G \rightarrow \mathbb{R}$ by $\chi(g) = b$ if $gt = a + bt$, and let $G^\rho = \{g \in G : |\chi(g)| \leq \rho\}$, $0 \leq \rho \leq \infty$. Thus $G^\rho \subset G$ is a normal sub-semigroup if $\rho \leq 1$. Everything that we discussed in this section applies to G^1 instead of G as well.

4. Riemannian manifolds. Let M be a complete Riemannian manifold, $\dim M = m > 0$, and N the manifold of its geodesics. The map $N \ni x \mapsto \dot{x}(0) \in TM$ identifies N and TM . We call this the canonical identification. Following Souriau's philosophy in [So], we shall mostly avoid using this identification, though; for any $t_0 \in \mathbb{R}$ the map $x \mapsto \dot{x}(t_0)$ would define just as natural an identification anyway. There is still a canonical symplectic form ω on N , see [W, 2.3]. Under any of the above identifications it corresponds to the form $\sum dq_i \wedge dp_i$ (written in terms of the usual local coordinates on $TM \approx T^*M$.) Note the sign convention used for ω .

Composition of geodesics with affine transformations $\mathbb{R} \rightarrow \mathbb{R}$ defines a right action of G on N . We shall consider adapted polarizations on G^1 -invariant open subsets $V \subset N$. The domains on which the adapted complex structures of [GS, LSz1] were defined correspond to the set of geodesics of speed $< r$, with $r \in (0, \infty]$ (but since then adapted complex structures on more general invariant sets have turned out to be of importance, see [FW] and references there). Put $\Omega_x(g) = A_g(x) = x \circ g$ for $x \in N, g \in G$. Then

$$A_g^* \omega = \chi(g) \omega. \quad (2)$$

When $gt = a + t$, (2) expresses the fact that the geodesic flow preserves ω , and when $gt = bt$, (2) holds because in the canonical identification $N \approx TM$ in local coordinates A_g is given by $(q_j, p_j) \mapsto (q_j, bp_j)$. Since G is generated by translations and dilations, (2) holds for all $g \in G$.

By Definition 1, given a polarization Q of G^1 , a polarization P of V is adapted to Q if $\Omega_x : (G^1, Q) \rightarrow (V, P)$ is polarized for every $x \in V$. If Q is left invariant and $0 < \rho < \infty$, this is equivalent to saying that $\Omega_x : (G^\rho, Q) \rightarrow (V, P)$ is polarized for every $x \in VG^{1/\rho}$.

Theorem 2. (a) *If a nonempty, G^1 -invariant open $V \subset N$ has a polarization P adapted to a polarization Q of G^1 , then Q is left invariant. If $Q = Q(s)$, $s \in \mathbb{C}$, then it determines P uniquely.*

(b) *If M is a closed analytic Riemannian manifold, then there is a G^1 -invariant open $V \subset N$, containing all zero speed geodesics, such that $VG^{1/|\operatorname{Im} s|}$ has a polarization P adapted to $(G^1, Q(s))$, for every $s \in \mathbb{C}$. The same is true if M is not necessarily closed, but modulo its isometry group it is compact.*

It is straightforward that, after the canonical identification $N \cong TM$, a polarization adapted to $Q(i)$ becomes the adapted complex structure of [LSz1, Definition 4.1].

Proof. Suppose (V, P) is adapted to Q . Fix a nonconstant geodesic $x \in V$; then $\Omega_x : (G^1, Q) \rightarrow (V, P)$ is a polarized immersion. Since $\Omega_{x \circ g} = \Omega_x \circ L_g$ is also

polarized for $g \in G^1$, it follows that $L_g: (G^1, Q) \rightarrow (G^1, Q)$ is polarized, i.e., Q is left invariant.

Let now $Q = Q(s)$, $s \in \mathbb{C}$. Uniqueness and existence of the adapted polarization for $s = i$ is the content of [LSz1, Theorem 4.2 and Sz1, Theorem 2.2]. To go further, with $g \in G$ and $x \in V$ consider the commutative diagram

$$\begin{array}{ccc} VG^{1/|\chi(g)|} & \xrightarrow{A_g} & V \\ \Omega_x \uparrow & & \uparrow \Omega_x \\ G^{1/|\chi(g)|} & \xrightarrow{R_g} & G^1, \end{array}$$

and recall that $R_g: (G, Q(gs)) \rightarrow (G, Q(s))$ is polarized. Now $A_g: VG^{1/|\chi(g)|} \rightarrow V$ is a diffeomorphism if $\chi(g) \neq 0$, and the diagram implies that A_g pulls back any $Q(gs)$ -adapted polarization to a $Q(s)$ -adapted polarization. Therefore uniqueness and existence for $s = i$ implies the same for any $s \in \mathbb{C} \setminus \mathbb{R}$. Concretely, if $gs = i$, then $\chi(g) = 1/\text{Im } s$, and so if V admits a $Q(i)$ -adapted polarization, then $VG^{1/|\text{Im } s|}$ will admit a $Q(s)$ -adapted polarization.

Finally, let $s \in \mathbb{R}$ and $\pi_s: N \rightarrow M$ be given by $\pi_s(x) = x(s)$. As said, the leaves of $Q(s)$ are left translates of the sub-semigroup $H_s \subset G$. That P is adapted to $Q(s)$ means it is tangential to the orbits $\Omega_x(H_s)$. As x ranges over a fiber $\pi_s^{-1}y$, these orbits all pass through the constant geodesic $\equiv y$, that we denote \bar{y} . Their tangents at \bar{y} form the vertical tangent space $T_{\bar{y}}(\pi_s^{-1}y)$, which then must agree with $P_{\bar{y}}$. Furthermore, the vector field generating the H_s -action, being tangent to the orbits, is a section of P . Since P is involutive, it must be invariant under the action of $H_s \cap G^1$. But π_s is also invariant, hence for any $x \in V$ and \bar{y} as above, $\pi_{s*}P_x = \pi_{s*}P_{\bar{y}} = 0$. Therefore P consists of the tangent spaces to the fibers of π_s , and is unique. It is straightforward that, conversely, the tangent spaces to the fibers form a polarization of N , adapted to $(G^1, Q(s))$ (and to $(G, Q(s))$).

The proof also gave the following

Corollary 3. *Let $s \in \mathbb{C}$ and $g \in G$. If a G^1 -invariant $V \subset N$ admits a $Q(s)$ -adapted polarization $P(s)$, then $VG^{1/|\chi(g)|}$ admits a $Q(gs)$ -adapted polarization $P(gs)$, and*

$$A_g: (VG^{1/|\chi(g)|}, P(gs)) \rightarrow (V, P(s))$$

is polarized, in fact a polarized isomorphism when $\chi(g) \neq 0$.

We shall continue using $P(s)$ for the $Q(s)$ -adapted polarization on $V \subset N$, whenever it exists. Let $v(x)$ denote the speed squared of a geodesic $x \in N$.

Theorem 4. *For $s \in \mathbb{C} \setminus \mathbb{R}$ let $\partial_s, \bar{\partial}_s$ denote the complex exterior derivations for the complex structure $P(s)$ on V (if this latter exists). The symplectic form ω on $V \subset N \cong TM$ is given by*

$$i\omega = (\text{Im } s)\bar{\partial}_s\partial_s v.$$

In particular, ω is a positive or negative $(1, 1)$ -form depending on whether $\text{Im } s > 0$ or < 0 .

Proof. When $s = i$, the claims are in [LSz1, Corollary 5.5 and Theorem 5.6]. (Note that E and Ω there correspond to our $v/2$, resp. $-\omega$.) Hence the general case follows by Corollary 3, because $A_g^*v = \chi(g)^2v$ and by (2), $A_g^*\omega = \chi(g)\omega$.

5. The canonical bundle. In the set up of Section 4, let $s \in \mathbb{C} \setminus \mathbb{R}$ and consider a G^1 -invariant open $V \subset N$ that admits a $Q(s)$ -adapted polarization $P(s)$, a complex structure. Its canonical bundle, the holomorphic line bundle $K \rightarrow V$ of $(m, 0)$ -forms, has a Hermitian metric h^K defined by

$$h^K(\theta)\omega^m(x) = i^{m^2} m! \theta \wedge \bar{\theta}, \quad \theta \in K_x, \quad x \in V. \quad (3)$$

In this subsection we compute h^K , something that is needed for purposes of quantization. We start by recalling certain constructions and results from [LSz1, Sz2]. Denote by $\text{id} \in G$ the identity transformation $\mathbb{R} \rightarrow \mathbb{R}$.

The action of G on N induces an action on TN and $\mathbb{C} \otimes TN$, denoted $(\xi, g) \mapsto \xi g$. Let $x \in V$. Any $\xi \in T_x V$ can be decomposed into $(1, 0)$ and $(0, 1)$ components with respect to the structure $P(s)$: $\xi = \xi^{1,0} + \xi^{0,1}$. If $J : TV \rightarrow TV$ denotes the complex structure operator for $P(s)$, then $\xi^{1,0} = (\xi - iJ\xi)/2$. The map $g \mapsto (\xi g)^{1,0}$ is holomorphic as a map $(G^1, Q(s)) \rightarrow T^{1,0}(V, P(s))$ (in the sense that it has a holomorphic extension to a neighborhood of G^1), see [LSz1, Proposition 5.1].

Now consider two m -tuples ξ_1, \dots and $\eta_1, \dots \in T_x N$, and assume that the $\xi_j^{1,0}$ are linearly independent. Those g for which $(\xi_j g)^{1,0}$ are linearly dependent form a discrete subset $\Delta \subset G$. For $g \in G^0 \setminus \Delta$ the $\xi_j g$ are also independent. Since when $g \in G^0$, the vectors $\xi_j g, \eta_j g$ are tangential to the m -dimensional manifold of zero speed geodesics, on $G^0 \setminus \Delta$ there is a smooth real $m \times m$ -matrix valued function $\phi^0 = (\phi_{jk}^0)$ such that

$$\eta_j g = \sum_k \phi_{jk}^0(g) \xi_k g, \quad g \in G^0 \setminus \Delta. \quad (4)$$

Further, there is a meromorphic $m \times m$ -matrix valued function $\phi = (\phi_{jk})$ on $(G^1, Q(s))$, with poles restricted to Δ , such that

$$(\eta_j g)^{1,0} = \sum_k \phi_{jk}(g) (\xi_k g)^{1,0}. \quad (5)$$

By (4) and (5), ϕ is the analytic continuation of ϕ^0 .

Theorem 5. *Suppose $x \in V$ and $\xi_1, \dots, \eta_m \in T_x V$ form a symplectic basis:*

$$\omega(\xi_j, \xi_k) = \omega(\eta_j, \eta_k) = 0, \quad \omega(\xi_j, \eta_k) = \delta_{jk}, \quad 1 \leq j, k \leq m.$$

If ϕ is as in (5), then for $\theta \in K_x$

$$h^K(\theta) = 2^m |\theta(\xi_1, \dots, \xi_m)|^2 \det \text{Im } \phi(\text{id}). \quad (6)$$

Proof. With $\zeta_j = \sum_k \text{Im } \phi_{jk}(\text{id}) \xi_k$

$$\begin{aligned} \theta \wedge \bar{\theta}(\zeta_1^{1,0}, \dots, \zeta_m^{1,0}, \eta_1^{0,1}, \dots, \eta_m^{0,1}) &= \theta(\zeta_1^{1,0}, \dots) \overline{\theta(\eta_1^{1,0}, \dots)} \\ &= \theta(\zeta_1, \dots) \overline{\theta(\eta_1^{1,0}, \dots)} = |\theta(\xi_1, \dots)|^2 \det \text{Im } \phi(\text{id}) \det \bar{\phi}(\text{id}). \end{aligned} \quad (7)$$

Taking real parts in (5) gives $\eta_j = \sum_k \operatorname{Re} \phi_{jk}(\operatorname{id}) \xi_k + J\zeta_j$. Thus

$$\begin{aligned} 2\omega(\zeta_j^{1,0}, \eta_l^{0,1}) &= 2\omega(\zeta_j^{1,0}, \eta_l) = \omega(\zeta_j - iJ\zeta_j, \eta_l) \\ &= \omega\left(\sum_k \operatorname{Im} \phi_{jk}(\operatorname{id}) \xi_k, \eta_l\right) - i\omega\left(\eta_j - \sum_k \operatorname{Re} \phi_{jk}(\operatorname{id}) \xi_k, \eta_l\right) \\ &= \operatorname{Im} \phi_{jl}(\operatorname{id}) + i\operatorname{Re} \phi_{jl}(\operatorname{id}) = i\bar{\phi}_{jl}(\operatorname{id}), \end{aligned}$$

and

$$\begin{aligned} \omega^m(\zeta_1^{1,0}, \dots, \zeta_m^{1,0}, \eta_1^{0,1}, \dots, \eta_m^{0,1}) &= m! i^{m(m-1)} \det(\omega(\zeta_j^{1,0}, \eta_l^{0,1})) \\ &= m! i^{m^2} 2^{-m} \det \bar{\phi}(\operatorname{id}). \end{aligned}$$

Comparing this with (3) and (7) yields (6).

6. The family of adapted polarizations. Finally we shall construct a polarized fibration $Z \rightarrow \mathbb{C}$ whose fibers represent the various $(V, P(s))$. With $s \in \mathbb{C}$, $x \in N$ consider the embeddings

$$i^x: \mathbb{C} \ni s \mapsto (s, x) \in \mathbb{C} \times N, \quad j^s: N \ni x \mapsto (s, x) \in \mathbb{C} \times N. \quad (8)$$

Also, let $\pi: \mathbb{C} \times N \rightarrow \mathbb{C}$ denote the projection.

Theorem 6. *Suppose that a G^1 -invariant open $V \subset N$ admits the adapted polarization (complex structure) $P(i)$.*

(a) On $Z = \{(s, x) \in \mathbb{C} \times N : x \in VG^{1/|\operatorname{Im} s|}\}$ there is a unique polarization P such that the maps

$$i^x: ((i^x)^{-1}Z, T^{1,0}\mathbb{C}) \rightarrow (Z, P), \quad j^s: (VG^{1/|\operatorname{Im} s|}, P(s)) \rightarrow (Z, P)$$

are polarized for all $s \in \mathbb{C}$, $x \in N$. With this P , $\pi: (Z, P) \rightarrow (\mathbb{C}, T^{1,0}\mathbb{C})$ is polarized, and $(Z \setminus \pi^{-1}\mathbb{R}, P) = Z_0$ is a complex manifold.

(b) Let $\partial, \bar{\partial}$ denote the complex exterior derivations on Z_0 , and $\tilde{\omega}$ the pullback of ω along the map $(s, x) \rightarrow x$. Then

$$i\tilde{\omega} = \bar{\partial}\partial(v\operatorname{Im} s) \quad \text{on } Z_0, \quad (9)$$

$v\operatorname{Im} s$ is plurisub/superharmonic if $\operatorname{Im} s > 0$, resp. < 0 , and satisfies the Monge–Ampère equation $\operatorname{rk} \bar{\partial}\partial(v\operatorname{Im} s) = m$.

(c) Finally, endow $(\mathbb{C}, T^{1,0}\mathbb{C}) \times (V, P(i))$ with the product complex structure. Then the map $\Phi: Z \rightarrow \mathbb{C} \times V$ given by

$$\Phi(s, x) = (s, x \circ g), \quad \text{where } gi = f_i(g) = s, \quad (10)$$

is polarized, and in fact restricts to a biholomorphism $Z_0 \rightarrow (\mathbb{C} \setminus \mathbb{R}) \times V$.

Proof. Since the range of i_*^x and j_*^s together span $T(\mathbb{C} \times N)$, the polarization P in question is unique, and must be given by

$$P_{(s,x)} = i_*^x T_s^{1,0}\mathbb{C} \oplus j_*^s P(s)_x, \quad (s, x) \in Z. \quad (11)$$

In view of Corollary 3 this formula defines a subbundle $P \subset \mathbb{C} \otimes TZ$. Our P has rank $m + 1$ all right, but is it involutive? To decide, first we check that Φ in (10) is polarized, i.e. Φ_* maps P into $T^{1,0}(\mathbb{C} \times V)$. With notation introduced earlier

$$(\Phi \circ i^x)(s) = (s, (\Omega_x \circ f_i^{-1})(s)), \quad (\Phi \circ j^s)(x) = (s, A_g x).$$

Now $\Omega_x: (G^r/\sqrt{v(x)}, Q(i)) \rightarrow (V, P(i))$ is holomorphic by the definition of $P(i)$ and by the observation preceding Theorem 2; also $f_i: (G, Q(i)) \rightarrow \mathbb{C}$ is biholomorphic by the definition of $Q(i)$. Therefore $\Phi \circ i^x$ is holomorphic. Similarly $\Phi \circ j^s: (VG^{1/|\text{Im } s|}, P(s)) \rightarrow (\mathbb{C} \times V, T^{1,0}(\mathbb{C} \times V))$ is polarized by Corollary 3 (s there corresponds to i here, though). Putting these and (11) together, we see $\Phi_* P \subset T^{1,0}(\mathbb{C} \times V)$ indeed.

Since $T^{1,0}(\mathbb{C} \times V)$ is involutive and Φ is a diffeomorphism over Z_0 , it follows that P is involutive over Z_0 , which therefore is a complex manifold. By density, P is involutive over all of Z . That π is polarized is obvious, so (a) and (c) have been proved. As to (9), the two sides restricted to the fibers of π agree by Theorem 4. Tangents to the fibers of $(s, x) \rightarrow x$, the “horizontal fibers”, constitute the kernel of $i\bar{\omega}$; to finish the proof it will suffice to show the same for $\text{Ker } \bar{\partial}\partial(v\text{Im } s)$. The restriction of $\bar{\partial}\partial(v\text{Im } s)$ to the horizontal fibers is certainly 0, since v restricts to a constant and i^x is holomorphic; but that is not quite enough. It will be necessary to compute $\bar{\partial}\partial(v\text{Im } s)$, that we do by pulling it back along Φ^{-1} .

If in (10) $gt = a + bt$, then $b = \text{Im } s$. Hence $(\Phi^{-1})^*(v\text{Im } s) = v/\text{Im } s$. (With a slight abuse of notation, v stands for both a function on N and its pull back along the projection $\mathbb{C} \times N \rightarrow N$. Also, $\text{Im } s$ stands for a function on $\mathbb{C} \times N$.) On $(\mathbb{C} \setminus \mathbb{R}) \times V$

$$\bar{\partial}\partial \frac{v}{\text{Im } s} = \frac{\bar{\partial}\partial v}{\text{Im } s} + \frac{d\bar{s} \wedge \partial v - \bar{\partial} v \wedge ds}{2i(\text{Im } s)^2} + \frac{v d\bar{s} \wedge ds}{2(\text{Im } s)^3}. \quad (12)$$

In computing ∂v , $\bar{\partial} v$, the operators corresponding to the complex structure $P(i)$ are to be used. From (12) $(\bar{\partial}\partial v/\text{Im } s)^{m+1}$ equals

$$\begin{aligned} (m+1) \left(\frac{\bar{\partial}\partial v}{\text{Im } s} \right)^m \wedge \frac{v d\bar{s} \wedge ds}{2(\text{Im } s)^3} - \binom{m+1}{2} \left(\frac{\bar{\partial}\partial v}{\text{Im } s} \right)^{m-1} \wedge \frac{(d\bar{s} \wedge \partial v - \bar{\partial} v \wedge ds)^2}{4(\text{Im } s)^4} \\ = \frac{(m+1)d\bar{s} \wedge ds}{4(\text{Im } s)^{m+3}} \wedge (2v(\bar{\partial}\partial v)^m - m(\bar{\partial}\partial v)^{m-1} \wedge \bar{\partial} v \wedge \partial v). \end{aligned}$$

By [LSz1, (5.10)], where $E = v/2$, the last expression vanishes. As $-i\bar{\partial}\partial(v/\text{Im } s)$ is definite along the fibers of π , its signature is m pluses (or minuses) and one 0, and the same holds for $-i\bar{\partial}\partial(v\text{Im } s)$ on Z_0 . In particular, $(v\text{Im } s)$ is plurisub/superharmonic, and because $\bar{\partial}\partial(v\text{Im } s)$ vanishes on the horizontal fibers, its kernel consists of the tangents to the horizontal fibers. This then proves (9) and the rest of (b).

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